Information Analysis and 2D Point Extrapolation using Method of Hurwitz-Radon Matrices

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Abstract: *Information analysis needs suitable methods of curve extrapolation. Proposed method of Hurwitz-Radon Matrices (MHR) can be used in extrapolation and interpolation of curves in the plane. For example quotations from the Stock Exchange, the market prices or rate of a currency form a curve. This paper contains the way of data anticipation and extrapolation via MHR method and decision making: to buy or not, to sell or not. Proposed method is based on a family of Hurwitz-Radon (HR) matrices. The matrices are skew-symmetric and possess columns composed of orthogonal vectors. The operator of Hurwitz-Radon (OHR), built from these matrices, is described. Two-dimensional information is represented by the set of curve points. It is shown how to create the orthogonal and discrete OHR and how to use it in a process of data foreseeing and extrapolation. MHR method is interpolating and extrapolating the curve point by point without using any formula or function*.

Keywords: *Information analysis, decision making, point interpolation, data extrapolation, value anticipation, hurwitz-radon matrices.*

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1. Introduction

A significant problem of information analysis and artificial intelligence [1] is that of appropriate data representation and extrapolation. Two-dimensional information can be treated as points on the curve [23]. Classical polynomial interpolation or extrapolation (Lagrange, Newton, Hermite) is useless for data anticipation, because the stock quotations or the market prices represent discrete information and they do not preserve a shape of the polynomial. Also Richardson extrapolation has some weak sides concerning discrete data. This paper is dealing with the method of value foreseeing by using a family of Hurwitz-Radon matrices. The quotations, prices or rate of a currency, represented by curve points, consist of information which allows us to extrapolate the next value and then to make a decision [5]. If the probabilities of possible actions are known, then some criteria are to apply: Laplace, Bayes, Wald, Hurwicz, Savage, Hodge-Lehmann [20] and others [24]. But author of this paper considers only two possibilities: to do something or not. For example to buy a share or not, to sell a currency or not. Proposed method of Hurwitz-Radon Matrices (MHR) is used in data extrapolation and then calculations for decision making are described. MHR method uses two-dimensional information for knowledge representation [14] and computational foundations [19]. Also medicine [18], industry and manufacturing are looking for the methods connected with geometry of the curves [21]. So suitable data representation and precise reconstruction or extrapolation [13] of the curve is a key factor in many applications of artificial intelligence [6, 12] and knowledge representation.

2. Information Representation

Information is represented by the set of curve points $(x_i,y_i)\in R^2$ (interpolation nodes) as follows in novel MHR method:

1. Nodes (characteristic points) are settled at local extrema (maximum or minimum) of one of coordinates and at least one point between two successive local extrema.
2. Nodes (x_i,y_i) are monotonic in coordinates x_i ($x_i<x_{i+1}$ for all i) or y_i ($y_i < y_{i+1}$).
3. One curve is represented by at least five nodes.

- *Condition* 1: is done for the most appropriate description of a curve. The quotations or prices are real coordinates of nodes.
- *Condition* 2: according to a graph of function means that x_i represent for example the time.
- *Condition* 3: is adequate to interpolation, but in extrapolation minimal number of nodes is four.

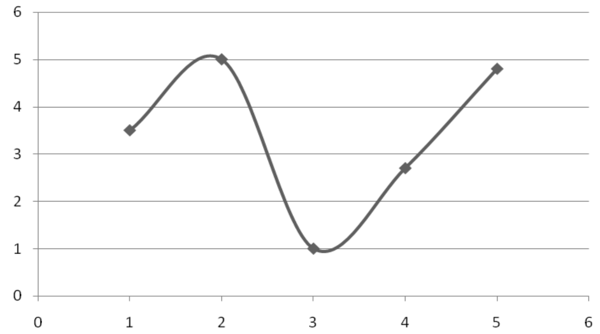

Figure 1. Five nodes of data and the curve.

Data points are treated as interpolation nodes. How can we extrapolate continues values at time x=5.5 for example or discrete data for next day x=6 (Figure 1)? The anticipation of values is possible using proposed MHR method.

3. Data Reconstruction

The following question is important in computer sciences and mathematics: is it possible to find a method of curve extrapolation in the plane without building the interpolation and extrapolation polynomials or other functions? This paper aims at giving the positive answer to this question. In comparison MHR method with Bézier curves, Hermite curves and B-curves (B-splines) or Non-Uniform Rational B-Spline (NURBS) one unpleasant feature of these curves must be mentioned: small change of one characteristic point can result in big change of whole reconstructed curve. Such a feature does not appear in MHR method. The methods of curve interpolation or extrapolation based on classical polynomial interpolations: Newton, Lagrange or Hermite polynomials and the spline curves which are piecewise polynomials [3]. Classical methods are useless to interpolate the function that fails to be differentiable at one point. Also when the graph of interpolated or extrapolated function differs from the shape of polynomials considerably, for example $f(x)=1/x$, interpolation and extrapolation is very hard because of existing local extrema of polynomial. Lagrange interpolation polynomial for function $f(x) = 1/x$ and nodes (5; 0.2), (5/3; 0.6), (1;1), (5/7;1.4), (5/9;1.8) has one minimum and two roots. Lagrange interpolation polynomial differs extremely from the shape of function $f(x)=1/x$.

We cannot forget about the Runge's phenomenon: when the interpolation nodes are equidistance then high-order polynomial oscillates toward the end of the interval, for example close to-1 and 1 with function $f(x) =1/(1+25x^2)$ and extrapolation is impossible [15]. MHR, described in this paper, is free of these bad examples. The curve or function in MHR method is parameterized [11] for real number $\alpha \in [0; 1]$ in the range of two successive interpolation nodes. MHR in 2D point extrapolation is possible with $\alpha < 0$ or $\alpha > 1$.

3.1. The Operator of Hurwitz-Radon

Adolf Hurwitz (1859-1919) and Johann Radon (1887-1956) published the papers about specific class of matrices in 1923, working on the problem of quadratic forms. Matrices A_i, $i = 1, 2\ldots m$ satisfying

$$A_jA_k+A_kA_j = 0,\ A_j^2 = -I \text{ for } j \neq k;\ j, k = 1, 2...m \tag{1}$$

are called *a family of Hurwitz-Radon matrices*. A family of Hurwitz-Radon (HR) matrices has important features [4]: HR matrices are skew-symmetric ($A_i^T = -A_i$) and reverse matrices are easy to find ($A_i^{-1} = -A_i$). Only for dimension N=2, 4 or 8 the family of HR matrices consists of N-1 matrices [12]. For N=2 there is one matrix:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2}$$

For N=4 there are three HR matrices with integer entries:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},\quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

For N = 8 we have seven HR matrices with elements 0, ±1. So far HR matrices are applied in electronics [2]: in Space-Time Block Coding (STBC) and orthogonal design [22], also in signal processing [15, 17] and Hamiltonian Neural Nets [16].

If one curve is described by a set of data points $\{(x_i, y_i), i = 1, 2, \ldots, n\}$ monotonic in coordinates x_i (time for example), then HR matrices combined with the identity matrix I_N are used to build the orthogonal and discrete Hurwitz-Radon Operator (OHR). For nodes (x_1, y_1) and (x_2, y_2), $x_1 < x_2$ OHR M of dimension N = 2 is constructed:

$$M = \frac{1}{x_1^2 + x_2^2}\begin{bmatrix} x_1y_1 + x_2y_2 & x_2y_1 - x_1y_2 \\ x_1y_2 - x_2y_1 & x_1y_1 + x_2y_2 \end{bmatrix} \tag{4}$$

For nodes (x_1, y_1), (x_2,y_2), (x_3,y_3) and (x_4,y_4), monotonic in x_i, OHR of dimension N = 4 is constructed:

$$M = \frac{1}{x_1^2 + x_2^2 + x_3^2 + x_4^2}\begin{bmatrix} u_0 & u_1 & u_2 & u_3 \\ -u_1 & u_0 & -u_3 & u_2 \\ -u_2 & u_3 & u_0 & -u_1 \\ -u_3 & -u_2 & u_1 & u_0 \end{bmatrix} \tag{5}$$

where

$$u_0 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4\,,$$
$$u_1 = -x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3\,,$$
$$u_2 = -x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2\,,$$
$$u_3 = -x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1\,.$$

For nodes (x_1, y_1), (x_2,y_2), …, (x_8,y_8), monotonic in x_i, OHR of dimension N = 8 is built [6] similarly:

$$M = \frac{1}{\sum_{i=1}^{8} x_i^2}\begin{bmatrix} u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \\ -u_1 & u_0 & u_3 & -u_2 & u_5 & -u_4 & -u_7 & u_6 \\ -u_2 & -u_3 & u_0 & u_1 & u_6 & u_7 & -u_4 & -u_5 \\ -u_3 & u_2 & -u_1 & u_0 & u_7 & -u_6 & u_5 & -u_4 \\ -u_4 & -u_5 & -u_6 & -u_7 & u_0 & u_1 & u_2 & u_3 \\ -u_5 & u_4 & -u_7 & u_6 & -u_1 & u_0 & -u_3 & u_2 \\ -u_6 & u_7 & u_4 & -u_5 & -u_2 & u_3 & u_0 & -u_1 \\ -u_7 & -u_6 & u_5 & u_4 & -u_3 & -u_2 & u_1 & u_0 \end{bmatrix} \quad (6)$$

where

$$u = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ -y_2 & y_1 & -y_4 & y_3 & -y_6 & y_5 & y_8 & -y_7 \\ -y_3 & y_4 & y_1 & -y_2 & -y_7 & -y_8 & y_5 & y_6 \\ -y_4 & -y_3 & y_2 & y_1 & -y_8 & y_7 & -y_6 & y_5 \\ -y_5 & y_6 & y_7 & y_8 & y_1 & -y_2 & -y_3 & -y_4 \\ -y_6 & -y_5 & y_8 & -y_7 & y_2 & y_1 & y_4 & -y_3 \\ -y_7 & -y_8 & -y_5 & y_6 & y_3 & -y_4 & y_1 & y_2 \\ -y_8 & y_7 & -y_6 & -y_5 & y_4 & y_3 & -y_2 & y_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \quad (7)$$

The components of the vector u = $(u_0, u_1, \ldots, u_7)^T$, appearing in the matrix M (3), are defined by (7) in the similar way but in terms of the coordinates of the above 8 nodes. Note that OHR operators M (4-6) satisfy the condition of interpolation

$$M \cdot \mathrm{x} = \mathrm{y} \quad (8)$$

for x = $(x_1, x_2 \ldots, x_N)^T \in R^N$, x $\neq 0$, y = $(y_1, y_2 \ldots, y_N)^T \in R^N$, N = 2, 4 or 8. If one curve is described by a set of nodes $\{(x_i, y_i), i = 1, 2, \ldots, n\}$ monotonic in coordinates y_i, then HR matrices combined with the identity matrix I_N are used to build the orthogonal and discrete reverse Hurwitz-Radon Operator (reverse OHR) M^{-1}. If matrix M is described as:

$$M = \frac{1}{\sum_{i=1}^{N} x_i^2}(u_0 \cdot I_N + D), \quad (9)$$

where matrix D consists of elements 0 (diagonal) and $u_1, \ldots, u_{N-1}$, then reverse OHR M^{-1} is given by:

$$M^{-1} = \frac{1}{\sum_{i=1}^{N} y_i^2}(u_0 \cdot I_N - D). \quad (10)$$

Note that reverse OHR operator (9) satisfies the condition of interpolation

$$M^{-1} \cdot \mathrm{y} = \mathrm{x} \quad (11)$$

for x = $(x_1, x_2 \ldots, x_N)^T \in R^N$, y = $(y_1, y_2 \ldots, y_N)^T \in R^N$, y $\neq 0$, N = 2, 4 or 8.

3.2. Point Extrapolation and MHR Method

Key question looks as follows: how can we compute coordinates of points settled between the interpolation nodes [7] or beyond the nodes? The answer is connected with proposed MHR method for interpolation [8] and extrapolation. On a segment of a line every number "c" situated between "a" and "b" is described by a linear (convex) combination $c = \alpha \cdot a + (1 - \alpha) \cdot b$ for

$$\alpha = \frac{b - c}{b - a} \in [0; 1] \quad (12)$$

If $c < a$ then $\alpha > 1$: possible extrapolation of points situated left of nodes. If $c > b$ then $\alpha < 0$ and possible extrapolation of points situated right of nodes. When the nodes are monotonic in coordinates x_i, the average OHR operator M_2 of dimension N=2, 4 or 8 is constructed as follows:

$$M_2 = \alpha \cdot M_0 + (1 - \alpha) \cdot M_1 \quad (13)$$

with the operator M_0 built (1)-(3) by "odd" nodes $(x_1 = a, y_1)$, (x_3, y_3), …, (x_{2N-1}, y_{2N-1}) and M_1 built by "even" nodes $(x_2 = b, y_2)$, (x_4, y_4), …, (x_{2N}, y_{2N}). Having the operator M_2 for coordinates $x_i < x_{i+1}$ it is possible to reconstruct the second coordinates of points (x,y) in terms of the vector C defined with

$$c_i = \alpha \cdot x_{2i-1} + (1 - \alpha) \cdot x_{2i},\ i = 1, 2, \ldots, N \quad (14)$$

as $C = [c_1, c_2, \ldots, c_N]^T$. The required formula is similar to (8):

$$Y(C) = M_2 \cdot C \quad (15)$$

in which components of vector $Y(C)$ give the second coordinate of the points (x,y) corresponding to the first coordinate, given in terms of components of the vector C. On the other hand, having the operator M_2^{-1} for coordinates $y_i < y_{i+1}$ it is possible to reconstruct the first coordinates of points (x,y):

$$M_2^{-1} = \alpha \cdot M_0^{-1} + (1 - \alpha) \cdot M_1^{-1},\ c_i = \alpha \cdot y_{2i-1} + (1 - \alpha) \cdot y_{2i},$$

$$X(C) = M_2^{-1} \cdot C \cdot \quad (16)$$

Calculation of unknown coordinates for curve points using (11)-(15) is called by author the method of Hurwitz-Radon Matrices (MHR) [9]. Here are Figure 2 the applications of MHR method for functions $f(x)=1/(1+25x^2)$ with five nodes equidistance in first coordinate: x_i = -1, -0.5, 0, 0.5, 1.

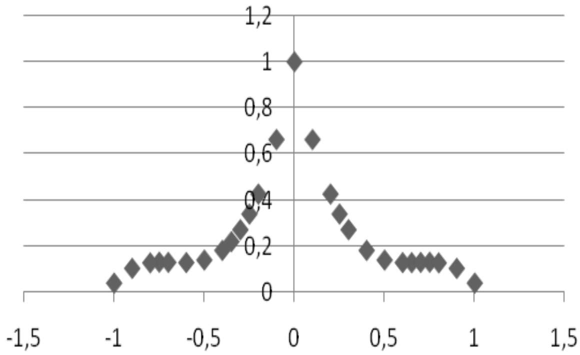

Figure 2. Twenty six interpolated points of functions $f(x)=1/(1+25x^2)$ using MHR method with 5 nodes.

MHR interpolation of function $f(x)=1/(1+25x^2)$ gives better result than Lagrange interpolation. The same can be said for function $f(x)=1/x$[10]. MHR extrapolation is valid for $\alpha < 0$ or $\alpha > 1$. In the case of continues information, parameter α is a real number. For example there are four nodes: (1; 2), (1.3;5), (2;3), (2.5;6). MHR extrapolation with α=-0.01 gives the point (2.505; 6.034) and with α=-0.1: (2.55; 6.348). But the rate of a currency or the quotations are discrete data. If we assume that the rate of a currency is represented by equidistance nodes (day by day -fixed step of time h=1 for coordinate x), next point or the rate on next day is extrapolated (anticipated) with α = -1.

3.3. Complexity of MHR Calculations

MHR interpolation of curve consists of L points: if we have n interpolation nodes, then there is $K = L–n$ points to find using MHR method. Now we consider the complexity of MHR calculations.

- Lemma 1. Let n=5, 9 or 17 is the number of interpolation nodes, let MHR method is done for reconstruction of the curve consists of L points. Then MHR method is connected with the computational cost of rank O(L).
- Proof. Using MHR method we have to reconstruct $K= L–n$ points of unknown curve. Counting the number of multiplications and divisions D here are the results:

1. $D = 4L+7$ for $n = 5$ and $L = 2i + 5$;
2. $D = 6L+21$ for $n = 9$ and $L = 4i + 9$;
3. $D = 10L+73$ for $n = 17$ and $L = 8i + 17$; i=2,3,4...□

The lowest computational cost appears in MHR method with five nodes and OHR operators of dimension $N = 2$.

4. Information Analysis

- *Example*: MHR calculations are done for true rates of euro at National Bank of Poland (NBP) from January 24th to February 14th, 2011. If last four rates are considered: (1; 3.8993), (2; 3.9248), (3; 3.9370) and (4; 3.9337), MHR extrapolation with matrices of dimension N=2 gives the result (5; 3.9158). So anticipated rate of euro on the day February 15th is 3.9158 (Figure 3).

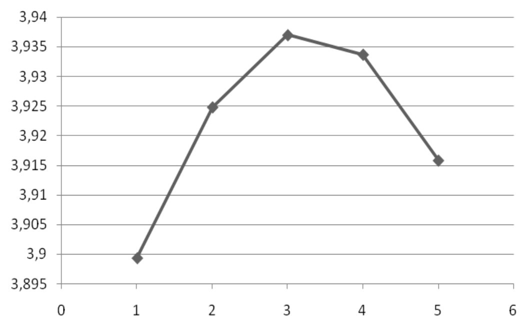

Figure 3. Extrapolated rate for day 5 (February 15th) using MHR method with 4 nodes.

If last eight rates are considered: (1;3.9173), (2;3.9075), (3;3.8684), (4;3.8742), (5;3.8993), (6;3.9248), (7;3.9370) and (8;3.9337), MHR extrapolation with matrices of dimension N=4 gives the result (9;4.0767). Anticipated rate of euro on the day February 15th is 4.0767 (Figure 4).

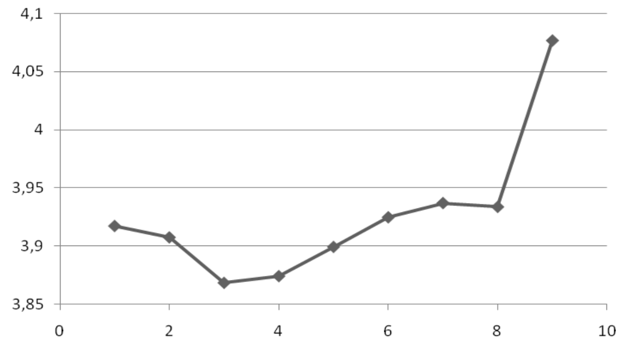

Figure 4. Extrapolated rate for day 9 (February 15th) using MHR method with 8 nodes.

There are two extrapolated values for next day. This example gives us two anticipated rates for tomorrow: 3.9158 and 4.0767, which differs considerably. How these extrapolated values can be used in the process of decision making: to buy euro or not, to sell euro or not? The proposal final anticipated rate of euro for the day February 15th (Figure 5) based on weighted mean value:

$$\frac{2 \cdot 3.9158 + 4.0767}{3} = 3.9694 \tag{17}$$

Because the rate 3.9158 is calculated for N=2, whereas 4.0767 is extrapolated for N=4. Formula (13) takes one fact into account: dimension N=4 is two times bigger than dimension N=2 and the result 3.9158 has to be strengthen multiplying by two.

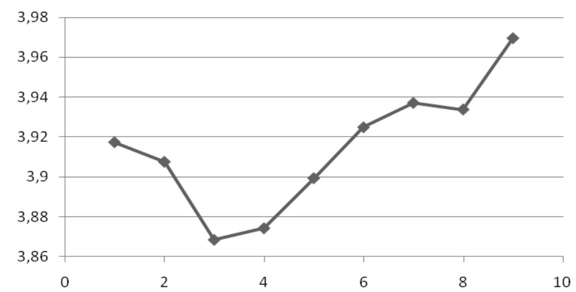

Figure 5. Extrapolated rate for day 9 (February 15th) using MHR method with 8 nodes and weighted mean value (13).

If last sixteen rates are considered, MHR extrapolation with matrices of dimension N=8 has to be used. Here are the rates: (1;3.8765), (2;3.8777), (3;3.8777), (4;3.9009), (5;3.9111), (6;3.9345), (7;3.9129), (8;3.9019), (9;3.9173), (10;3.9075), (11;3.8684), (12;3.8742), (13;3.8993), (14;3.9248), (15;3.9370) and (16;3.9337). Average OHR operator M_2 and MHR calculations look as follows:

$$\begin{bmatrix} 0.3226 & 0.0154 & 0.0286 & 0.0462 & -0.0620 & 0.0444 & 0.0924 & 0.0461 \\ -0.0154 & 0.3226 & 0.0462 & -0.0286 & 0.0444 & 0.0620 & -0.0461 & 0.0924 \\ -0.0286 & -0.0462 & 0.3226 & 0.0154 & 0.0924 & 0.0461 & 0.0620 & -0.0444 \\ -0.0462 & 0.0286 & -0.0154 & 0.3226 & 0.0461 & -0.0924 & 0.0444 & 0.0620 \\ 0.0620 & -0.0444 & -0.0924 & -0.0461 & 0.3226 & 0.0154 & 0.0286 & 0.0462 \\ -0.0444 & -0.0620 & -0.0461 & 0.0924 & -0.0154 & 0.3226 & -0.0462 & 0.0286 \\ -0.0924 & 0.0461 & 0.0620 & -0.0444 & -0.0286 & 0.0462 & 0.3226 & -0.0154 \\ -0.0461 & -0.0924 & 0.0444 & -0.0620 & -0.0462 & -0.0286 & 0.0154 & 0.3226 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ 7 \\ 9 \\ 11 \\ 13 \\ 15 \\ 17 \end{bmatrix} = \begin{bmatrix} 3.7252 \\ 3.8072 \\ 3.8704 \\ 3.8278 \\ 3.8653 \\ 3.8834 \\ 3.9825 \\ 3.9882 \end{bmatrix} \quad (18)$$

MHR extrapolation gives the result (17; 3.9882). Anticipated rate of euro for the day February 15th is 3.9882 (Figure 6).

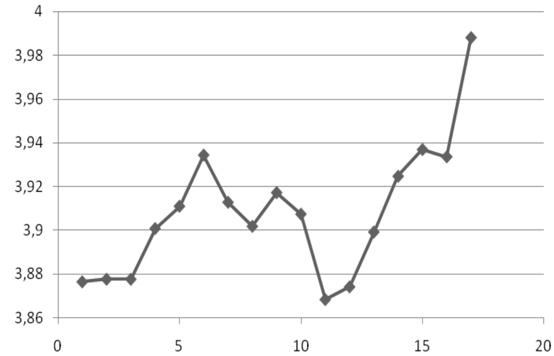

Figure 6. Extrapolated rate for day 17 (February 15th) using MHR method with 16 nodes.

MHR extrapolation has been done for three times (N = 2, 4 or 8) and anticipated values are 3.9158, 4.0767 and 3.9882 respectively. The proposal final anticipated rate of euro for the day February 15th (Figure7) based on weighted mean value:

$$\frac{4 \cdot 3.9158 + 2 \cdot 4.0767 + 3.9882}{7} = 3.9721 \quad (19)$$

because the rate 3.9158 is calculated with last four data points, 4.0767 is extrapolated for last eight information points and 3.9882 is computed for last sixteen data points. Formula (18) takes one fact into account: number of sixteen points is four times bigger than four and two times bigger than eigth. The result 3.9158 has to be strengthen multiplying by four and the rate 4.0767 has to be strengthen multiplying by two.

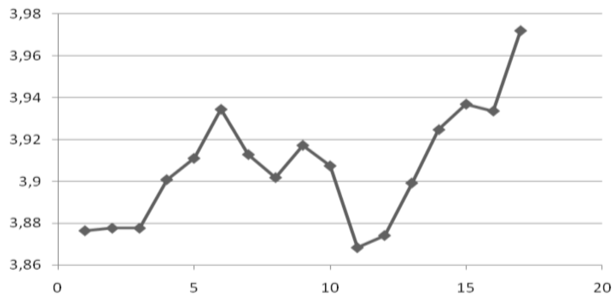

Figure 7. Extrapolated rate for day 17 (February 15th) using MHR method with 16 nodes and weighted mean value (14).

The true rate of euro for the day February 15th is 3.9398 (Figure 8).

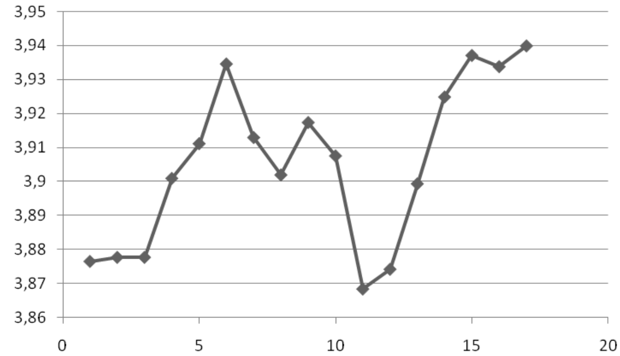

Figure 8. The true rate of euro for day 17 (February 15th).

In author's opinion, values extrapolated for next day 3.9694 (13) and 3.9721 (14) are good enough to be one of the factors for making a decision of buying or selling the currency.

5. Conclusions

The method of Hurwitz-Radon Matrices leads to curve interpolation and value extrapolation depending on the number and location of information points. No characteristic features of curve are important in MHR method: failing to be differentiable at any point, the Runge's phenomenon or differences from the shape of polynomials. These features are very significant for classical polynomial interpolations and extrapolations. MHR method gives the possibility of reconstruction a curve and anticipation the data points. The only condition is to have a set of nodes according to assumptions in MHR method. Information representation and curve extrapolation by MHR method is connected with possibility of changing the nodes coordinates and reconstruction of new data or curve for new set of nodes. The same MHR interpolation and extrapolation is valid for discrete and continues information. Main features of MHR method are: accuracy of data reconstruction depending on number of nodes; interpolation or extrapolation of a curve consists of L points is connected with the computational cost of rank $O(L)$; MHR method is dealing with local operators: average OHR operators are built by successive 4, 8 or 16 points, what is connected with smaller computational costs then using all nodes; MHR is not an affine interpolation. Future works are connected with: possibility to apply MHR method to three-dimensional curves (3D data), computing the extrapolation error, object recognition and MHR version for equidistance nodes.

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