Operator Decomposition of Graphs

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Abstract: *In this paper we introduce a new form of decomposition of graphs, the (P, Q)-decomposition. We first give an optimal algorithm for finding the 1-decomposition of a graph which is a special case of the (P, Q)-decomposition which was first introduced in [21]. We then examine the connections between the 1-decomposition and well known forms of decomposition of graphs, namely, modular and homogeneous decomposition. The characterization of graphs totally decomposable by 1-decomposition is also given. The last part of our paper is devoted to a generalization of the 1-decomposition. We first show that some basic properties of modular decomposition can be extended in a new form of decomposition of graphs that we called operator decomposition. We introduce the notion of a (P, Q)-module, where P and Q are hereditary graph-theoretic properties, the notion of a (P, Q)-split graph and the closed hereditary class (P, Q) of graphs (P and Q are closed under the operations of join of graphs and disjoint union of graphs, respectively). On this base, we construct a special case of the operator decomposition that is called (P, Q)-decomposition. Such decomposition is uniquely determined by an arbitrary minimal nontrivial (P, Q)-module in G. In particular, if G $\not\in$ (P, Q), then G has the unique canonical (P, Q)-decomposition.*

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1. Introduction

All graphs considered are finite, undirected, without loops and multiple edges. For all notions not defined here the reader is refered to [3]. The vertex and the edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively, while n denotes the cardinality of $V(G)$ and m the cardinality of $E(G)$. We write $u \sim v \left(u \not\sim v\right)$ if vertices u and v are adjacent (non-adjacent). For the subsets $U, W \subseteq V(G)$ the notation $U \sim W$ means that $u \sim w$ for all vertices $u \in U$ and $w \in W$, $U \not\sim W$ means that there are no adjacent vertices $u \in U$ and $w \in W$. To shorten notation, we write $u \sim W$ ($u \not\sim W$) instead of $\{u\} \sim W$ ($\{u\} \not\sim W$). The subgraph of G induced by a set $A \subseteq V(G)$ is denoted by $G[A]$. We write $\overline{G}$ for the complement graph of G. The neighborhood of a vertex v in the graph G is denoted by $N_G^{(v)}$ (or $N(v)$), $\overline{N}_G(v) = V(G) \setminus v \setminus N_G(v)$.

One type of graph decomposition based on the well-known notion of split graphs is investigated. A triad $T = (G, A, B)$, where G is a graph and (A, B) is an ordered bipartition of $V(G)$ into a clique A and a stable set B, is considered as an operator acting on the set of graphs. An operator T acts on a graph H by formula:

$$TH = G \cup H \cup \{ax / a \in A, x \in V(H)\} \qquad (1)$$

(all edges of the complete bipartite graph with the parts A and $V(H)$ are added to the disjoint union $G \cup H$).

An isomorphism of triads is defined as an isomorphism of 2-colored graphs. Denote by Tr the set of triads distinguished up to isomorphism of triads. The action (1) induces the associative binary operation on Tr. So the set Tr becomes a semigroup of operators with the exact action on the set of graphs. The semigroup Tr was introduced in [21]. The following structure theorem of the decomposition was presented in the same paper.

A graph F is called decomposable if there exist a triad T and a graph H such that $F = TH$, otherwise F is indecomposable. The decomposition theorem asserts that every decomposable graph F can be uniquely represented in the form:

$$F = T_1 T_2 \ldots T_k F_0$$

Where T_i is indecomposable element of the semigroup Tr and F_0 is indecomposable graph. This theorem occurs to be useful instrument for the characterization and enumeration of several graph classes [19, 22]. On the base of the theorem, the Kelly-Ulam reconstruction conjecture was proved for the class of decomposable graphs. A criterium of decomposability of graphs was presented in [23]. In the same paper on the base of the decomposition theorem an exhaustive description of unigraphs was obtained. (A graph is called a unigraph if it is determined uniquely up to isomorphism by its degree sequence). Namely, it was proved that a graph is a unigraph if and only if all its indecomposable components are unigraphs, and the catalogue of indecomposable unigraphs was given.

In this paper, a decomposition theory is developed. In section 2, we present the 1-decomposition which was first introduced in [21]. An *o* (*n*) algorithm of constructing 1-decomposition starting from the degree sequence of the vertices of a graph is presented in section 3. In section 4, several examples of the applications of 1-decomposition are discussed concerning the structure and recognition of some classes of graphs. We study also a connection between the 1-decomposition and two well known forms of decomposition of graphs namely modular and homogeneous decomposition. Finally, we characterize by forbidden subgraphs the family of graphs which are totally decomposable by the 1-decomposition. A far-reaching notion of a more general decomposition that we called (*P*, *Q*)-decomposition is introduced in section 5. We conclude this section by giving an example of the application of (*P*, *Q*)-decomposition.

2. Basic Structure Theorem of 1-Decomposition

This decomposition is based on the well-known notion of split graphs. A graph G is called split [6] if there is a partition of its vertex set into a clique *A* and an independent set *B*. We call this partition a bipartition. One of the parts can be empty, but not both.

$$V(G) = A \cup B \tag{2}$$

In what follows, a sequence of length *n* is called an *n-sequence*. The *i*th member of a sequence *d* is denoted by d_i. An *n*-sequence *d* is *graphical* if a graph exists (a *realization* of the sequence *d*) such that *d* is its degree sequence (the list of its vertex degrees). A graphical sequence is called *split* if it has a split realization. A splitness criterium for a graph is formulated in terms of vertex degrees. Therefore, all realizations of a split sequence are split. An n-sequence d is called proper if

$$n-1 \geq d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$$

Obviously, a graphical sequence can be assumed to be proper.

Theorem 1: For a proper graphical n-sequence d put $m = m(d) = max\{i : d_i \geq i - 1\}$. The sequence d is split if and only if the following equality holds

$$\sum_{i=1}^{m} di = m(m-1) + \sum_{i=m+1}^{n} di \tag{3}$$

For m = 1 and m = n equality (3) has the form:

$$\sum_{i=1}^{n} di = 0 \; or \; \sum_{i=1}^{n} di = n(n-1),$$

respectively [12].

It is convenient to consider split graphs together with fixed bipartitions. For a split graph *G* with bipartition (2), we shall call the triad (*G*, *A*, *B*) a splitting of *G* or a splitted graph.

Theorem 1 implies that a proper split sequence *d* can be divided into two parts d^A and d_B which are the lists of vertex degrees for the upper (*A*) and the lower (*B*) parts of its realizations, respectively (one of the parts can be empty). The sequense *d* written in the form:

$$d = (d^A; d^B)$$

is called a *splitting* of *d* or a *splitted sequence*. A splitted graph having d^A and d_B as the lists of vertex degrees for its upper and lower parts, respectively, is called a *realization* of the splitted sequence d.
The following assertion is obvious.

Corollary 1: A proper split n-sequence d with $d_{m(d)} \rangle m(d) - 1$ has the unique splitting

$$(d_1, \ldots, d_{m(d)+1}, \ldots d_n) \tag{4}$$

If $d_{m(d)} = m(d) - 1$, the sequence d has exactly two splittings, namely,(4) and

$$(d_1, \ldots, d_{m(d)-1}; d_{m(d)}, \ldots, d_n).$$

The concept of isomorphism of splitted graphs appears naturally. Let (*G*, *A*, *B*) and (*H*, *C*, *D*) be two splitted graphs, and $f : G \rightarrow H$ be a graph isomorphism. If *f* preserves the parts, i. e., $f(A) = C$ and $f(B) = D$, then *f* is called an isomorphism of splitted graphs (*G*, *A*, *B*) and (*H*, *C*, *D*). In this case we write $f : (G, A, B) \rightarrow (H, C, D)$ and (*G*, *A*, *B*) ≅ (*H*, *C*, *D*). It may happen that $(G, A, B) \ncong (H, C, D)$, although $G \cong H$ (for example, two splitted graphs resulting from $K_4 - e$).

In what follows, graphs are considered up to isomorphism, but splitted ones are considered up to isomorphism of splitted graphs. Denote by Σ and Γ the sets of splitted graphs and of simple graphs, respectively. Define the composition $\circ : \Sigma \times \Gamma \rightarrow \Gamma$ composition as follows:

If $\sigma \in \Sigma, \sigma = (G, A, B), H \in \Gamma$, then

$$\sigma \circ H = (G \cup H) + \{av : a \in A, v \in V(H)\} \tag{5}$$

The edge set of the complete bipartite graph with the parts *A* and *V* (*H*) is added to the disjoint union $G \cup H$ If, in addition, *H* is a split graph with a bipartition $V(H) = C \cup D$, then the composition $\sigma \circ H = F$ is split as well with the bipartition $V(f) = (A \cup C) \cup (B \cup D)$. In this case we suppose

$$(G, A, B) \circ (H, C, D) = (F, A \cup C, B \cup D) \tag{6}$$

In what follows we omit the sign $\circ$ of the composition.
Formula (6) defines a binary algebraic operation on the set Σ of triads which is called the multiplication of

triads. It is clear that this operation is associative. In what follows Σ is regarded as a semigroup with the multiplication (6).

Formula (5) defines an action of the semigroup Σ on the set of graphs, i. e.,

$$(\sigma\rho)\ G = \sigma\ (\rho G) \text{ for } \sigma, \rho \in \Sigma, G \in \Gamma.$$

We call a representation of a graph G in a form:

$$G = \sigma_1 \ldots \sigma_k H, \quad \sigma_i \in \Sigma, H \in \Gamma$$

an operator decomposition of the graph G.

An element $\sigma \in \Sigma$ is called decomposable if there are $\alpha, \beta \in \Sigma$ such that $\sigma = \alpha\ \beta$. Otherwise σ is indecomposable. Analogously, a graph G is called 1-decomposable (or decomposable on the level split) if $G = \sigma H, \sigma \in \Sigma, H \in \Gamma$. Otherwise G is 1-indecomposable (indecomposable on the level split).

Theorem 2:

1. An n-vertex graph G with a proper degree sequence

   $$d = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n,$$

   is 1-decomposable if and only if there exist nonnegative integers p and q such that

   $$0 < p + q < n, \sum_{i=1}^{p} d_i = p(n - q - 1) + \sum_{i=n-q+1}^{n} d_i \quad (7)$$

2. Call a pair (p, q) satisfying the conditions (7) good. One can associate with every good pair (p, q) the decomposition

   $$G = (F, A, B)\ H \quad (8)$$

   Where

   $$(d_1, \ldots, d_p), (d_{p+1} \ldots, d_{n-q}), \ and\ (d_{n-q+1}, \ldots, d_n)$$

   are the vertex degree lists from A, $V(H)$, and B, respectively. Moreover, every 1-decomposition of the form (8) is associated with some good pair.
3. Let p_o be the minimum of the first components in good pairs, and $q_0 = |\{i\text{: } d_i < p_0\}|$ if $p_0 \neq 0$ and $q_0 = 1$ for $p_0 = 0$ then the triad (G, A, B) in (8) is indecomposable if and only if the relevant good pair (p, q) coincides with (p_0, q_0) [23].

Corollary 2: The component H in operator decomposition of the form $G = \sigma H$ is 1-indecomposable if and only if for the associated good pair (p, q) the parameters p and q are the maximums of the first and the second coordinates in good pairs, respectively.

Theorem 3:

1. Every graph G can be represented as a composition

   $$G = (G_1, A_1, B_1) \ldots (G_k, A_k, B_k)\ G_0 \quad (9)$$

   of indecomposable components. Here (G_i, A_i, B_i) are indecomposable splitted graphs and G_0 is an 1-indecomposable graph. (If G is 1-indecomposable, then there are no splitted components in (9)). (Decomposition (9) is called 1-decomposition of G).
2. Graphs G and G' with 1-decompositions (9) and

   $$G' = (G'_1, A'_1, B'_1) \ldots (G'_l, A'_l, B'_l) G'_0$$

   are isomorphic if and only if the following conditions hold:

   a. $G_0 \cong G'_0$
   b. $k = l$
   c. $(G_i, A_i, B_i) \cong (G'_i, A'_i, B'_i)$, $i = 1 \ldots, k$

   Denote by Σ*and Γ^* the sets of indecomposable elements in the semigroup Σ and of 1-indecomposable graphs, respectively [21].

By the decomposition theorem, each element σ in the semigroup Σ of splitted graphs can be uniquely decomposed into the product:

$$\sigma = \sigma_1 \ldots \sigma_k, k \geq 1, \sigma_i \in \Sigma^*$$

and every decomposable graph G can be uniquely represented as the decomposition:

$$G = \sigma G_0, \sigma \in \Sigma, G_0 \in \Gamma^*$$

of the operator part σ and the indecomposable part G_0, (we call $V(G_0)$ a 1-module). In other words, the following corollary holds.

Corollary 3: The set Σ of splitted graphs is the free semigroup over the alphabet Σ* with respect to multiplication (6). A free action of this semigroup is defined by (5).

The triads $T_i = (G_i, A_i, B_i)$ with $B_i = \phi$ are allowed in decomposition (9). We call such triads *A*-parts of *G*. It is abvious that $|A_i| = 1$ in every *A*-part. We call two *A*-parts A_i and A_j, $i < j$, *undivided* if every T_k, $i < k < j$, is A-part also. *Undevided B*-parts ($A_i = \phi$) are defined analogously. If one substitutes all undevided *A*-parts as well as undevided *B*-parts by their products, one gets a canonical 1-decomposition of *G*. This decomposition does not contain neighboring *A*-parts (as well as neighboring *B*-parts). Theorem 3 implies that canonical 1-decomposition of a graph *G* is determined uniquely.

3. An *O* (*n*) Algorithm for Constructing the Canonical 1-Decomposition

The algorithm is based on Theorem 3. The set of vertices of a graph *G* with equal degrees is called a link in G.

Corollary 4: For every graph all the vertices of one link are included in the same canonical component. If the component is a triad, then a link is contained entirely in the upper or in the lower part of the triad.

Theorem 4: The canonical 1-decomposition of a graph can be constructed in time O (n) from the degree sequence of the graph.

Proof: Here we assume that vertices of an arbitrary graph G are enumerated in the proper order, i. e., $\deg v_i \geq \deg v_{i+1}$, $i = 1,\ldots, n - 1$. We denote by:

$$C = (C_1^{k_1}, C_2^{k_2}, \ldots, C_N^{k_N}), C_i \neq C_j \qquad (10)$$

the brief degree sequence of G. Here, N is a number of pairwise different vertex degrees of G, and k_i is a number of vertices of G with the degree D_i, $i = 1, \ldots, N$. Let

$$D = (D_1, D_2, \ldots, D_N) \qquad (11)$$

be corresponding sequence of links in G, i. e., $D_i = \{v \in V(G) : \deg v = C_i\}$. Note that k_i is the number of vertices in D_i. We give a description of an algorithm constructing the canonical 1-decomposition of G.

Input: The degree sequence (10) of a graph G and the corresponding sequence (11).

Output: Triads $T_i = (G\,[A_i \bigcup B_i], A_i, B_i)$ and a set $M \subseteq V(H)$ such that $G = T_1T_2 \ldots T_rG\,[M]$ is the canonical 1-decomposition of G.

It follows from Corollary 4 that we can regard A_i (B_i) as the set of indices of members of D, which are included in the upper (lower) part of the component T_i. Analogously, we consider M as a set of indices of corresponding members of D.

Step 0: Construct the sequences $S = (S_0, \ldots, S_N)$ and $K = (K_0, \ldots, K_N)$ of the sums as follows:

$S_0 := 0, K_0 := 0, \quad S_i := S_{i-1} + C_ik_i, K_i := K_{i-1} + k_i, i = 1,\ldots, N$.

The sequences S and K can be constructed in time $O(N)$. For arbitrary $n_0 \leq n_1 \leq r$ we have

$$\sum_{v \in \bigcup_{n_0 \leq i \leq n_1} D_i} deg\, v = S_{n_1} - S_{n_0 - 1}, \sum_{j=n_0}^{n_1} \left|D_j\right| = K_{n_1} - K_{n_0-1}.$$

Let the components $T_1, \ldots, T_{i-1}$ of the decomposition be already constructed. We denote by f_D (l_D) the minimal (maximal) index of members of D that are contained in none of $T_1, T_2, \ldots, T_{i-1}$. Let f (l) be minimal (maximal) of the vertex indices in D_{f_D} (D_{l_D}). Initially, when $i = 1$, put $f_D = 1, f = 1$; $l_D = N$, $l = n$.

Step 1: Dominating sets.

If $C_{f_D} = l - 1$, then:

1.1 construct T_i as follows: $A_i := \{f_D\}$, $B_i := \{0\}$;

1.2 $f_D := f_D + 1, f := f + k_{f_D}$.

Step 2: Isolated sets.

If $C_{l_D} = f - 1$, then:

2.1 construct T_i as follows: $A_i := \{0\}$, $B_i := \{l_D\}$;

2.2 $l_D := l_D - 1, l := l - k_{lD}$.

Step 3: We will denote by p_D, q_D the sought for $|A_i|$, $|B_i|$, respectively, and by p, q the numbers of vertices in the upper and the lower parts of the relevant triad T_i (i. e., $p = |\bigcup_{j \in A_I} D_j|$, $q = |\bigcup_{j \in B_I} D_j|$).

3.0 Put $p_D = 1$, $p = k_{f_D}$, Bound=0.

3.1 Find a natural number q_D corresponding to p_D, that is a number satisfying the following conditions:

(cl) Bound< $p_D + q_D < l_D - f_D + 1$;

($l_D - f_D + 1$ naturally is the number of elements D_i that are contained in none of $T_1, T_2, \ldots, T_{i-1}$)

(c2) $C_{l_D + q_D + 1} < p + (f - 1)$.

(c3) $C_{l_D - q_D} \geq p + (f - 1)$.

If such q_D does not exist, then go to (5) (the number r of triads in the decomposition equals to i - 1).

3.2 $q := k_{l_D} - K_{l_D - q_D}$.

Step 3 requires $O(q_D$ - Bound) time.

Step 4: Check whether the pair (p_D, q_D) is good. If the equation

$$(S_{f_D + p_D - 1} - S_{f_D - 1}) - p(f - 1) = p(l - f - q + (S_{l_D} - S_{l_D - q_D}) - q(f - 1)$$

holds, then the pair (p_D, q_D) is good and perform:

4.1 construct T_i as follows:

$A_i := \{f_D, f_D + 1, \ldots, f_D + p_D - 1\}$,
$B_i = \{l_D - {}_D + 1, l_D - q_d + 2, \ldots, l_D\}$;

4.2 $f_D := f_D + p_D$, $f := f + p$,
$I_D := I_D - q_D$, $1 := 1 - q$.

Otherwise perform:

4.3 $p := p + k_{f_D + p_D}$, $p_D := p_D + 1$;

4.4 Bound := q_D. Go to (3.1).

Step 5: M := { $f_D + f_D + 1, \ldots, l_D$}.

Condition (cl) and operation on Bound in (4.4) provide that finding one component T_i requires $O(q_D + P_D)$ time, where $p_D = |A_i|$, $q_D = |B_i|$. Since after constructing T_i we actually diminish in (2.2), (4.2) the unprocessed sequence D (i.e. we diminish the boundary in (cl)) on $p_D + q_D$, the total complexity of the algorithm is $O(N)$. □

4. Applications of Structure Theorem of 1-Decomposition

4.1. Characterization and Recognition of Some Graph Classes

Let us consider several simple examples. An edge set $E' \subseteq EG$ is independent in G if the subgraph of G induced by E' is a threshold graph. Let I_E denote the family of independent edge sets. Analogously, a vertex set $V' \subseteq V(G)$ is independent if $G[V']$ is a threshold graph. Let I_E denotes the family of independent vertex sets. A graph G is matroidal [17] if the independence system (E, I_E) is a matroid. A graph G is matrogenic [7] if the independence system (V, I_V) is a matroid.

Let G be an arbitrary graph. Define a binary relation $\geq$ on $V(G)$ by $u \geq v \Leftrightarrow N[u] \supseteq N[v]$, this relation is a preorder and it is called the vicinal preorder of G. If $u \geq v$ or $v \geq u$, then u and v are called comparable. Otherwise, u and v are incomparable, we denote this fact by $u \| v$. A preorder $\geq$is total if u $u \geq v$ or $v \geq u$ for every $u, v \in V(G)$.

A graph G is called *box-threshod (BT)* [18] if for every two vertices $u, v \in V(G)$ satisfy $u \| v \Rightarrow \deg x = \deg y$, or equivalently:

$$\deg u \rangle \deg v \Rightarrow u \succ v.$$

A graph G is called *regular* if all its vertex degrees are equal. A split graph G with bipartition (A, B) is *biregular* if all vertices from A have the same degree and all vertices from B have the same degree.

A $(2K_2, C_4)$ free graph is called a *pseudo-split* [2]. A detailed description of the class *PSplit* of pseudo-split graphs is presented in [2]. In [16] the linear-time recognizes whether a graph is pseudo-split is proved. We repeat the last result on the base of 1-decomposition.

It is well known that the class (1, 1) of split graphs coincides with the class of $\{2K_2, C_4, C_5\}$ free graphs [6], so $(1, 1) \subset$ PSplit. Let $G \in$ PSplit $\setminus (1, 1)$. Then G contains an induced subgraph $H \cong C_5$. Obviously if a vertex v of $G - V(H)$ is adjacent (not adjacent) to some vertex of H then it is adjacent (non adjacent) to every vertex of H and all such vertices form a clique (stable set) in the graph G. Therefore $G = TC_5$ for $G \neq C_5$ and $T \in (1,1)Tr$.

A graph G is called a *net* [14] if its vertex set $V(G)$ can be partitioned into the sets K and S such that:

1. K is a clique, S is a stable set, and $|K| = |S| \geq 2$.
2. There exists a bijection f between K and S such that either $N(x) = \{f(x)\}$ for all vertices x in S (a thin net) or else $N(x) = K - \{f(x)\}$ for all vertices x in S (a thick net).

The 1-decomposition theory allowed to describe the structure and enumerated matroidal and matrogenic graphs [22], box-threshold graphs [19].

Theorem 5: A graph is threshold, matroidal, matrogenic, or box-threshold if and only if all its indecomposable components are threshold, matroidal, matrogenic, or box-threshold, respectively.

Theorem 6: Let G be a graph, and (9) be its 1-decomposition.

1. The graph G is matroidal if and only if [22]:
   - All its 1-indecomposable components G_i, $1 \leq i \leq k$, are one-vertex graphs or nets.
   - The last component G_0 is one-vertex, net, perfect matching of more than one edge, or the complements of this matching.
2. The graph G is matrogenic if and only if [22]:
   - All its 1-indecomposable components G_i, $1 \leq i \leq k$, are one-vertex graphs or nets.
   - The graph G_0 is one-vertex, net, perfect matching of more than one edge, the complements of this matching or the chordless pentagon C_5.
3. The graph G is box-threshold if and only if [19]:
   - All its 1-indecomposable components G_i $1 \leq i \leq k$, are split biregular graphs.
   - The graph G_0 is split biregular or non-split regular graph.
4. The graph G is pseudo-split if and only if G_0 is split or $G_0 \cong C_5$

Corollary 5: Matroidal graph, matrogenic graph, box-threshold graph, and pseudo-split graph can be recognized in linear time starting from its degree sequence.

4.2. Connection of 1-Decomposition with Modular Decomposition

In this section we shall adapt the construction of the well known modular decomposition tree for taking into account the 1-decomposition of a graph. Let us remind some notions and facts from the modular decomposition theory (see, e. g., [3]).

Let G be a graph, $M \subseteq V(G)$. M is called a module of G if $v \sim M$ or $v \nsim M$. For every vertex $v \in V(G) \setminus M$. If M is a module, then $V(G)$ is naturally partitioned into three parts:

$$V(G) = A \cup B \cup M, A \sim M, B \nsim M. \quad (12)$$

The partition (12) is associated with the module M.

For every graph G, the sets $V(G)$, singleton subsets of $V(G)$ and Ø are modules. These modules are called *trivial*. All the other modules are *nontrivial*. A graph is *prime* if it contains only trivial modules.

Two modules M', M are *overlapping* if the sets $M' \cap M, M' \backslash M, M \setminus M'$ are all nonempty. A module M of a graph G is called strong if for any other module M' the modules M, M' do not overlap, i.e. one of the following conditions holds:

$$M' \cap M = \phi \; M' \subseteq M, M' \supseteq M.$$

For example, the vertex sets of the connected components of the graph G, the vertex sets of the connected components of the complement graph $\overline{G}$, as well as trivial modules are strong modules.

We write $M_1 < M_2$ ($M_1 \lhd M_2$) if M_1 is a module (strong module) of the graph $G[M_2]$, $M_2 \subseteq V(G)$. One can immediately check the following.

Proposition 1: The binary relations $<$ and $\lhd$ are transitive.

A maximal strong submodule H of a module M is a strong module of G strictly contained in M, such that every strong module strictly containing H, contains also M. Proposition 1 directly implies.

Proposition 2: The maximal strong submodules of a strong module M of a graph G are exactly all maximal with respect to inclusion strong modules of G [M].

For an arbitrary graph G we define the directed graph $T(G)$: The vertices (nodes) of $T(G)$ are in a bijective correspondence with the strong modules of the graph G and have the names of these modules; the vertex M_2 is a son of the vertex M_1 if and only if M_2 is a maximal strong submodule of M_1.

Theorem 7: For any graph G the graph T (G) is a rooted tree with a root in the vertex V (G).

Proof: It is sufficient to prove that every vertex of the graph $T(G)$ has only one parent. Conversely, let the vertex M has two parents: M_1 and M_2. So $M \subset M_1$ and $M \subset M_2$ The definition of a strong module implies that either $M_1 \subset M_2$ or $M_2 \subset M_1$. In any case we have the contradiction with the maximality of the strong module M.

□

Let us mark every vertex of the tree $T(G)$ by one of the labels

$$P, S, N, l \tag{13}$$

as follows:

- P if the induced subgraph $G[M]$ is not connected.
- S if the induced subgraph $\overline{G}(M)$ is not connected.
- N if |M|>1 and both induced subgraphs G [M] and $\overline{G}(M)$ are connected.
- l if |M| = 1.

The labeled tree $T(G)$ is called the modular decomposition of the graph G. Linear time algorithms for the modular decomposition were presented in [4, 5].

Often the tree $T(G)$ is defined recursively. The application of 1-decomposition in such a recursion is based on the following theorem.

Theorem 8: Let G be biconnected (both G and $\overline{G}$ are connected) 1-decomposable graph and $G = TH$ be the 1-decomposition such that $T = (G, A_1, B_1)$ is an indecomposable triad. Let $(A_1^1, A_1^2, \ldots, A_1^r)$ be a partition of A_1 into subsets of vertices with equal neighborhoods in B_1, and let $(B_1^1, B_1^2, \ldots, B_1^s)$ be analogous partition of B_1 into subsets of vertices with equal neighborhoods in A_1. Put $M = V(H)$. Then

$$M, A_1^1, A_1^2, \ldots, A_1^r, B_1^1, B_1^2, \ldots, B_1^s$$

is the list of maximal strong modules of the graph G.

Proof: Let M' be a non-trivial module in G and

$$M' \not\subseteq M. \tag{14}$$

Put

$$\widetilde{A}_1 = A_1 \setminus M', \; \widetilde{B}_1 = B_1 \setminus M', \; \widetilde{A}_2 = A_1 \cap M',$$
$$\widetilde{B}_2 = B_1 \cap M', C = M \cap M'.$$

Let $C \neq \phi$ then since $\widetilde{A}_1 \sim C$ and $\widetilde{B}_1 \nsim C$, we have $\widetilde{A}_1 \sim \widetilde{B}_2$ and $\widetilde{B}_1 \nsim \widetilde{A}_2$. From (14) we have $\widetilde{A}_2 \cup \widetilde{B}_2 \neq \phi$ and therefore $\widetilde{A}_1 \cup \widetilde{B}_1 = \phi$, otherwise the triad T is decomposable:

$$T = (G_1, A_1, B_1) = (G_1(\widetilde{A}_1 \cup \widetilde{B}_1), \widetilde{A}_1, \widetilde{B}_1)(G_1(\widetilde{A}_2 \cup \widetilde{B}_2), \widetilde{A}_2, \widetilde{B}_2).$$

So $M' \supset A_1 \cup B_1$. Since the graph G is biconnected, then both sets A_1 and B_1 are nonempty. Further we have $M \sim A_1$, $M \nsim B$ and therefore $M' = V(G)$. But M' is nontrivial. The contradiction obtained proves that condition (14) implies

$$M' \cap M = \phi. \tag{15}$$

We proved that M is maximal strong module in G.

If now M' is maximal strong module in G distinct from M, then (15) holds and so $M' \subseteq A_1$ or $M' \subseteq B_1$. The further proof follows from the definition of a module. □

So if a graph G is 1-decomposable and (9) is its 1-decomposition, then the modular decomposition tree of G has the form represented in Figure 1.

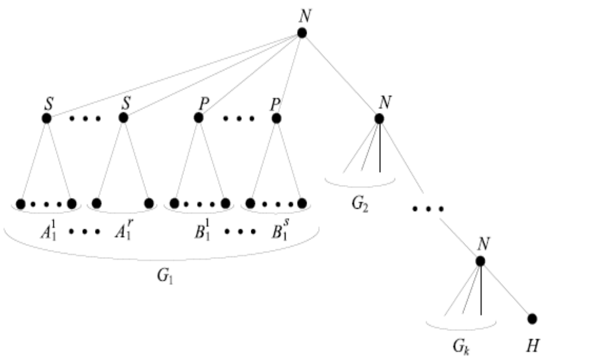

Figure 1. The modular decomposition tree of an 1-decomposable graph.

Now, define the tree T_1 (G) from the tree T (G) by replacing the subtrees corresponding to the triads (G_i, A_i, B_i) by leaves with appropriate names (G_i, A_i, B_i), and changing the label N of the fathers of (G_i, A_i, B_i) by the label 1.

Note that by Theorem 8 one can easily transform the tree T_1 (G) into T (G) if it is necessary. But for some problems it is sufficient to have T_1 (G) only, and we do not need to apply a "heavy artillery" of constructing N-nodes in modular decomposition tree. Let us present some examples of such cases.

A graph G is a spider if it can be represented in the form G = (H, K, S) R where H is a net with its net partition (K, S), and R is an arbitrary graph. In other words a spider G is a graph of the one of the following types:

a. G is split graph with a bipartition $V(G) = K \cup S$ such that all edges between K and S form a perfect matching.
b. $G = \overline{H}$, where H is a graph of type (a).
c. $G = (H, K, S)R$ or $G = (\overline{H}, K, S)R$, where H is a graph of type (a) and R is an arbitrary graph.

Proposition 3: A graph G is a spider if and only if the first indecomposable component in its 1-decomposition is of the form (a) or (b). Hence it can be easily recognized from its degree sequence.

A graph G is called P_4-sparse [10] if no set of five vertices induces more than one P_4 in G.

Theorem 9: A graph G is P_4-sparse if and only if one of the following conditions holds for every induced subgraph H of G with at least two vertices [14]:

1. H is disconnected.
2. $\overline{H}$ is disconnected.
3. H is isomorphic to a spider.

A graph G is called P_4-reducible [13] if no vertex in G belongs to more than one induced P_4 of G. Clearly the class of P_4-reducible graphs is a subclass of P_4-sparse graphs.

In our terms the characterization theorem of P_4-reducible graphs is formulated in the following way.

Theorem 10: A graph G is P_4-reducible if and only if for every induced subgraph H of G exactly one of the following conditions is satisfied [13]:

1. H is disconnected.
2. $\overline{H}$ is disconnected.
3. H can be represented in the form $H = (H_1, A_1, B_1)$ H_0, $H_1 \cong P_4$.

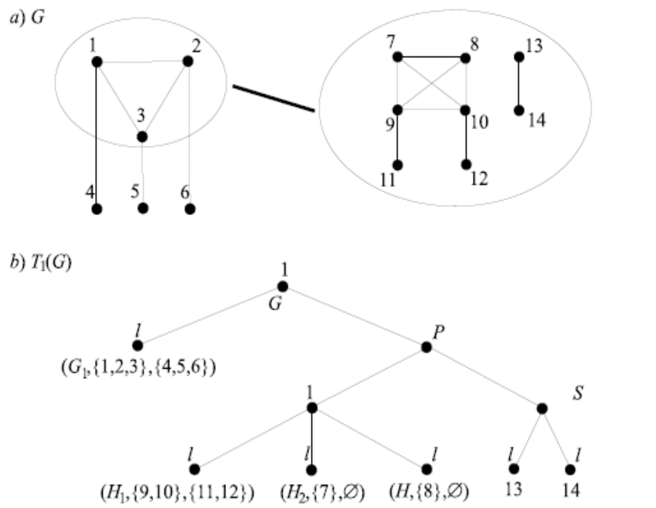

Figure 2. An example of a P_4-sparse graph G and its tree $T_1(G)$.

Note that a spider is *prime* if $|R| \leq 1$ (in this case the spider contains only trivial strong modules).

Theorem 11:

1. A graph G is P_4-sparse if and only if T_1 (G) does not contain an N-node and its leaf is either a clique or a stable set or a prime spider.
2. A graph G is P_4-reducible if and only if T_1 (G) does not contain an N-node and its leaf is either a clique or a stable set or a P_4.

Corollary 6:

1. The set of P_4-sparse graphs is a closure of the set of the one-vertex graph and prime spiders with respect to the operations of disjoint union of graphs, join of graphs, and the multiplication on the prime spiders.
2. The set of P_4-reducible graphs is a closure of the set of one-vertex graph and P_4 with respect to the operation of disjoint union of graphs, join of graphs, and the multiplication on P_4.

So, the tree T_1 (G) of the P_4-sparse and P_4-reducible graph G does not contain an N-node and *we* don't need to find strong modules for recognizing P_4-sparse and P_4-reducible graphs.

4.3. Totally 1-Decomposable Graphs

Denote by (1, P, S) the class of graphs whose decomposition tree T_1 (G) does not contain an N-node (totally T_1-decomposable graphs). Now, we are going to characterize the class (1, P, S).

The endpoints of the P_4 with the vertex set {a, b, c, d} and the edge set {ab, bc, cd} are the vertices a and d while b and c are the midpoints of this P_4. Let G be an arbitrary graph and let q be an internal node of T (G). We denote by M (q) the corresponding module of G i. e., the set of leaves of the subtree in T (G) with the root in the vertex q. Let also $V(q) = \{q_1, q_2, ..., q_r\}$ be the set of sons of q in T (G). The *representative* graph G (q) of the module M (q) is the graph whose vertex set is V (q) and $q_i \sim q_j$ if and only if there exists a vertex of M (q_i) that is adjacent to a vertex of M (q_i), Note that by definition of a module, if a vertex of M (q_i) is adjacent to a vertex of M (q_j), then $M(q_i) \sim M(q_j)$. Thus, G (q) is isomorphic to the subgraph of G induced by a subset of M (q) consisting of a single vertex from each maximal strong submodule of M (q) in the modular decomposition of G. So we can define a representative graph G (q) as the induced subgraph $G[x_1, x_2, ..., x_r]$ of a graph G where x_i is an arbitrary vertex in M (q_i), i = 1, 2, . . . , r.

It is easy to see that if q is an S-node then G (q) is a complete graph, if q is a P-node then G (q) is edgeless, and if q is an N-node then G (q) is a prime graph. Let $Z_{(1,\ P,\ S)} = \{Z_1, Z_2, Z_3, Z_4, Z_5\}$ as shown in Figure 3. Now, on the base of modular decomposition theory and two following known theorems we are going to show that (1, P, S) is exactly the class of $Z_{(1,\ P,\ S)}$-free graphs.

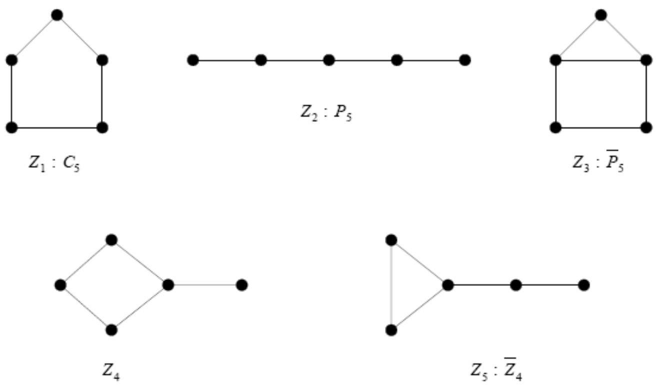

Figure 3. The set $Z_{(1,\ P,\ S)}$ of graphs.

Theorem 12: Every prime graph G containing C_4 contains Z_3 (a house) or Z_6 (a domino) or a graph Z_7 as shown in Figure 4 [11].

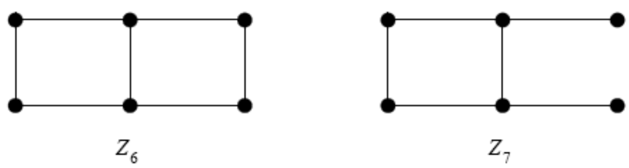

Figure 4. The graphs for Theorem 12.

It is evident, that for every prime split graph G there exists a bipartition $(K \cup R, B)$ such that $K \cup R$ is a clique, B is a stable set, |R| ≤ 1, $deg_B(v) = 0$ for a vertex $v \in R$ and $deg_B(u) > 0$ for every vertex $u \in K$.

Theorem 13: Let G be a prime split graph with a bipartition $(K \cup R, B)$ then every vertex of B is an endpoint of an induced P_4 of G and every vertex of K is a midpoint of an induced P_4 in G [8].

Lemma 1: Every prime $Z_{(1,\ p,\ s)}$-free graph G is split.

Proof: Note that a domino Z_6 and a graph Z_7 both contain the graph Z_4 from $Z_{(1,\ p,\ s)}$, as well as a complement graph $\overline{Z}_6$ and a complement graph $\overline{Z}_7$ both contain $Z_5 = \overline{Z}_4$. So, Theorem 12 together with the characterization of split graphs in terms of forbidden induced subgraphs prove the lemma. □

Lemma 2: Let G be a $Z_{(1,\ P,\ S)}$-free graph, and α be an N-node of the modular decomposition tree T (G). Then *G (α)* is a split graph with a bipartition $(K \cup R, B)$ verifying the following conditions:

1. M (x) is a stable set for every node $x \in B$.
2. M (y) is a clique for every node $y \in K$.

Proof: Obviously, the representative graph *G (α)* is prime and, by Lemma 1, it is split. By Theorem 13, x is an endpoint of an induced $P_4 = G_{[x,\ y1,\ y2,\ x2]}$ of the graph *G (α)*. If M (x) contains an edge u_1u_2, then the induced subgraph $G_{[ul,\ u2,\ y1,\ y2,\ x2]}$ is exactly Z_5, a contradiction.

Let $G_{[xl,\ y,\ y3,\ x3]}$ be an induced P_4 of the graph G (α) where y is a midpoint according to Theorem 13. If $M_{(y)}$ contains two non adjacent vertices v_1 and v_2, then the graph G contains a subgraph $G_{[x1,\ vl,\ v2,\ y3,\ x3]}$ isomorphic to Z_4. The contradiction obtained proved the theorem.

□

Theorem 14: A graph G is T_1 -totally decomposable if and only if it is $Z_{(l,\ p,\ s)}$ free.

Proof: The "if" part is evident since every graph in $Z_{(1,\ p,\ s)}$ is biconnected and 1-indecomposable. Now, let G be $Z_{(1,\ p,\ s)}$-free. By Lemma 2, for every N-node α in T (G) the graph M (α) is either split (when R = ø) or 1-decomposable (where M (v) is the 1-module for a vertex v from nonempty R). Therefore the tree T_1 (G) does not contain an N-node.

4.4. P-Connected Graphs and 1-Decomposition

A graph G is called P_4-connected [15] (or P-connected) if for every partition of V (G) into nonempty disjoint sets V_1 and V_2 there exists a crossing P_4, that is a P_4 containing vertices from both V_1 and V_2. The P-connected components (P-components) of a graph are the maximal induced P-connected subgraphs (or its vertex sets). Obviously P-connected components are onevertex graphs or have at least four vertices.

A P-connected graph G is separable if its vertex set $V(G)$ can be partitioned into two nonempty disjoint sets V_1 and V_2 in such a way that every crossing P_4 has its midpoint in V_1 and it endpoints in V_2. The partition (V_1, V_2) is called a separation. The following theorem provides the foundation of the homogeneous decomposition of graphs.

Theorem 15: For an arbitrary graph G exactly one of the following conditions is satisfied [15]:

1. Is disconnected.
2. $\overline{G}$ is disconnected.
3. There is a unique proper separable P-connected components S of G with a partition (S_1, S_2) such that every vertex outside S is adjacent to all vertices in S_1 and to no vertex in S_2.
4. G is P-connected.

Now, we are going to outline the connection between 1-decomposition and homogeneous decomposition.

Lemma 3: If G is a split 1-indecomposable graph, then G is P-connected.

Proof: Let (A, B) be the bipartition of $V(G)$. Assume that G is not P-connected and let

$$V_1 \cup V_2 \cup ... \cup V_1 = V(G)$$

be the partition of V (G) into P-components, and let

$$(A_i, B_i) = (A \cap V_i, B \cap V_i), i = 1,2,...,l.$$

Fact 1: If there exist two adjacent vertices $x \sim y$, $x \in A_i$, $y \in B_j$, *then* $y \sim A_i$.

Proof: Without loss of generality assume that $|V_i| \neq 1$. Then $|V_i| \geq 4$. There exists a P_4 in V_1 containing x. Let

$$G[b_1, a_1 x, b_2] \cong P_4, a_1 \not\sim b_2, b_1, b_2 \in B_i, a_1, x \in A_i.$$

It is evident that $a_1 \sim y$, otherwise $G[y, x, a_1, b_1] \cong P_4$. Let H be another P_4 in V_i, such that $\{b_1, a_1, x, a_2\} \cap V(H) \neq \phi$. If at least one a_1 or x is a midpoint of H, then y is adjacent to both midpoints of H. Now assume that $b_2 \in V(H)$. Let a_3 be a midpoint of H adjacent to b_2. We have $y \sim a_3$ otherwise $G_{[b2, a3, a1, y]} \cong P_4$. Further y is adjacent to both midpoints of H also.

Let $a_k \in A_i$. Since V_i is P-connected, then there exists a sequence $H_1, H_2, ..., H_r$ of induced subgraphs of G, such that

$$V(H_j) \subseteq V_i, H_j \cong P_4, j = 1, 2, ..., r,$$

$V(H_j) \cap V(H_{j+1}) \neq \phi, j = 1, 2, ..., r-1, x \in V(H_1), a_k \in V(H_r)$

Since $y \sim x$, y is adjacent to both midpoints of every H_j, in particular $y \sim a_k$. □

Fact 2: *If* $y \sim A_i$ *for some vertex* $y \in B_j$, *then* $A_i \sim B_j$.

Proof: On the contrary, suppose that there exist two non-adjacent vertices $a_i \not\sim b_j, a_i \in A_i, b_j \in B_j$. Then in complement graph $\overline{G}$ the conditions of Fact 1 hold for vertices a_i and b_j. So we have $a_i \not\sim B_j$, in particular $a_i \not\sim y$, a contradiction. □

Fact 3: *If* $A_i \sim B_j, B_i \neq \phi, A_j \neq \phi$, *then* $B_i \not\sim A_j$.

Proof: Suppose, contrary to our claim, that there exist two adjacent vertices $b_i \sim a_j, b_i \in B_i, a_j \in A_j$. Then $B_i \sim A_j$ by Fact 1 and Fact 2.

Let:

$$G[b_1, a_1, a_2, b_i] \cong P_4, a_1, a_2 \in A_i,\ b_1, b_i \in B_i, a_1 \not\sim b_i,$$
$$G[b_2, a_j, a_3, b_3] \cong P_4, a_j, a_3 \in A_j, b_2, b_3 \in B_j, b_2 \not\sim a_3.$$

Obviously, $G[b_2, a_1, a_3, b_i] \cong P_4$, which is impossible since V_i and V_j are p-components. □

Fact 4: *If* $B_i \not\sim A_j, A_i \neq \phi, B_j \neq \phi$ *then* $A_i \sim B_j$.

Suppose that G contains an induced subgraph P_4. So, there exists a p-component V_i such that $A_i \neq \phi$ and $B_i \neq \phi$. Denote by $N^A(i)$ the set of p-components V_j, $j \neq i$, such that $A_i \sim B_j$. Denote by $\overline{N}_{B(i)}$ the set of p-components V_j, $j \neq i$, such that $B_i \not\sim A_j$. Note that by *Fact 3* and *Fact 4*, if $V_j \in N^A$ and $A_j \neq \phi$, then $V_j \in \overline{N}_B$. Analogously if $V_j \in \overline{N}_B$ and $B_j \neq \varphi$, then $V_j \in N^A$. If $M = N^A(i) \cup \overline{N}_B(i) = \phi$, then, obviously, V_i is 1-module.

Fact 5: If $M \neq \phi$, then M is 1-module.

Proof: Conversely, suppose that there exists two p components $V_j \in M$ and $V_K \notin M$ containing two non-adjacent vertices $a_k \not\sim b_j, a_k \in A_K, b_j \in B_j$. Let $a_i \not\sim b_i$ be two non-adjacent vertices in V_i. We have $a_i \sim b_j, a_k \sim b_i$ and therefore $G[b_i, a_k, a_i, b_j] \cong P_4$, a contradiction. One can obtain analogous contradiction if suppose that there exists two p-components $V_j \in M$ and $V_k \notin M$ containing two adjacent vertices $b_k \sim a_j, b_k \in B_k, a_j \in A_j$. □

If G does not contain a P_4 (G is cograph), then G contains either a dominating vertex a or an isolated vertex b. In any case G is 1-decomposable, a contradiction. □

Theorem 16: If G is a graph and (9) is its 1-decomposition, then $G_1, G_2, ..., G_k$ are p-components nets of G.

Remark 1: If G is 1-decomposable graph and (9) is its 1-decomposition, then the P-component S with the bipartition (S_1, S_2) from condition (3) of Theorem 15 is exactly V (G_l) with the bipartition (A_1, B_1).

5. Generalization of 1-Decomposition

5.1. Semigroup of Triads

The notion of a module suggests the idea to insert an arbitrary graph G_0 as a module in a graph G, that is to consider an operation $(G, A, B)\ G_0 = G \bigcup G_0 + \{ab : a \in A, b \in V(G_0)\}$ where (A, B) is an arbitrary bipartition of the vertex set $V(G)$. (In the graph $(G, A, B)\ G_0$ obtained the set $V(G_0)$ is a module and the partition $(V(G_0), A, B)$ is associated with the module $V(G_0)$. Obviously, one can obtain different graphs $(G, A, B)\ G_0$ from the same graphs G and G_0 taking different bipartitions $V(G) = A \bigcup B$.

Consider triads $T = (G, A, B)$ where G is a graph and (A, B) is an ordered partition of the set $V(G)$ into two disjoint subsets (a bipartition). The sets A and B are called the upper and the lower parts of the graph G (triad T), respectively (one of the parts can be empty).

Let $T_i = (G_i, A_i, B_i)$, $i = 1, 2$, be two triads. An isomorphism $\beta : V(G_1) \to V(G_2)$ of the graphs G_1 and G_2 preserving the bipartition ($\beta(A_1) = A_2, \beta(B_1) = B_2$)) is called an isomorphism of triads $T_1 \to T_2$. We write $T_1 \ncong T_2$ if and only if there exists an isomorphism $T_1 \to T_2$. Clearly, the situation when $G_1 \cong G_2$ but $T_1 \ncong T_2$ is possible even when $|A_1| = |A_2|$.

Denote the set of all triads (graphs) up to isomorphism of triads (graphs) by T_r (G_r). We consider the triads from T_r as left operators acting on the set G_r, the action of the operators is defined by the formula

$$(H, A, B)G = G \bigcup H + \{ax : a \in A, x \in V(G)\} \quad (16)$$

So on the set T_r the action (16) induces a binary algebraic operation (the multiplication of triads):

$$(G_1, A_1, B_1)(G_2, A_2, B_2) = ((G_1, A_1, B_1)G_2, A_1 \cup A_2, B_1 \cup B_2) \quad (17)$$

Lemma 4: The set T_r is a semigroup with respect to the multiplication (17). Formula (16) determines the action of T_r on G_r. In other words, T_r is a semigroup of operators on G_r, i. e.,

$$(T_1T_2)T_3 = T_1(T_2T_3),\ (T_1T_2)G = T_1(T_2G)$$

for all

$$T_i \in T_r, i = 1,2,3\ G \in G_r.$$

The following statement contains a number of simple properties of modules that were mentioned several times by different authors (see, for example, [3]).

Lemma 5: For an arbitrary graph G the following statements (1-4) are true:

1. If M_1 and M_2 are modules, then $M_1 \bigcap M_2$ is a module.
2. If M_1 and M_2 are modules, $M_1 \bigcap M_2 \neq \phi$, then $M_1 \bigcup M_2$ is a module.
3. If M_l is a module of G, M_2 is a module of G $[M_l]$, then M_2 is a module of G.
4. If M is a module of G, $U \subset V(G)$, then $U \bigcap M$ is a module of G [U].

Lemma 5 can be naturally modified into the corresponding statement for the triads by replacing the words "graph G" and "module" with the words "triad T" and "T-module", respectively. A triad T is called decomposable if it can be represented as a product of two triads. Otherwise, it is indecomposable (or prime).

Let $T = (G, A, B) \in T_r$, M be a module in G and $(M, \widetilde{A}, \widetilde{B})$ be the associated partition. We call M a T-module if

$$\widetilde{A} \subseteq A, \widetilde{B} \subseteq B$$

It is evident that an arbitrary singleton module is not necessary T-module. Therefore it is reasonable to consider singleton T-modules to be nontrivial.

Lemma 6: A triad $T = (G, A, B)$ is decomposable if and only if there exists a nontrivial T-module in G.

Proof: Obviously, *if $T = (G_1, A_1, B_1)(G_2, A_2, B_2)$, then $A_2 \bigcup B_2$ is a nontrivial T-module of* G. On the other hand let M be a nontrivial T-module of G, (A_1, B_1, M) be the associated partition, $A_2 = A \setminus A_1$, $B_2 = B \setminus B_1$, $G_1 = G\lfloor A_1 \cup B_1 \rfloor$, $G_2 = G[M]$, $T_1 = (G_1, A_1, B_1)$, $T_2 = (G_2, A_2, B_2)$. Then we have (16)

$$T = T_1T_2 \quad (18)$$

□

The module M and the decomposition (18) are said to be *associated.* Modified Lemma 5 together with Lemma 5 implies Lemma 7.

Lemma 7: Let (17)

$$T = (G, A, B) = T_1T_2 \quad (19)$$

And let M be T-module of G associated with decomposition (19). Then:

1. The triad T_1 is indecomposable if and only if M is a maximal (with respect to inclusion) nontrivial T-module of G.
2. The triad T_2 is indecomposable if and only if M is a minimal (with respect to inclusion) nontrivial T-module of G.
3. If $A' \subseteq A$, $B' \subseteq B$, $V = A' \bigcup B'$, $I = M \cap V$ and $T' = (G[V], A', B')$ then I is a T'-module of G [V].

It is evident that every triad T can be represented as a product:

$$T = T_1 T_2 ... T_k, \ k \geq 1 \tag{20}$$

of indecomposable triads T_i. We call such representation a decomposition of T into *indecomposable parts*.

An indecomposable part T_i with empty lower (upper) part is called an *A-part* (a *B-part*). A-parts T_i and T_j, $i < j$, are called *undivided* if every indecomposable part T_k, $i < k < j$, is an A-part also. *Undivided* B-parts are defined analogously.

Theorem 17: The decomposition of a triad into indecomposable parts is determined uniquely up to permutation of undivided A-parts or undivided B-parts.

Proof: The statement is obvious for indecomposable triads. Further apply induction on the number of vertices.
Let

$$T = T_1 T_2 ... T_k, T = T_1' T_2' ... T_l', k, l > 1$$

be two decompositions of triad T into indecomposable parts

$$(G_1, A_1, B_1) = T_1 \neq T_1' = (G_1', A_1', B_1')$$

Putting $T_2 ... T_k = S = (H, C, D)$ and $T_2' ... T_l' = (H', C', D')$ we have

$$T = T_1 S, T = T_1' S'.$$

By Lemma 7, the sets $M = C \bigcup D$ and $M = C' \bigcup D'$ are maximal T-modules, and the intersestion $M' \bigcap (A_1 \bigcup B_1) = I$ is a T_1-module of $G[A_1 \bigcup B_1]$. Now Lemma 6 implies that the module I is trivial. On the other hand, $I = \phi$ since M' is maximal. So $I = A_1 \bigcup B_1$ and therefore (by symmetry)

$$A_1 \bigcup B_1 \subseteq M', A_1' \bigcup B_1' \subseteq M \tag{21}$$

By (21), the triad T_1 is the first indecomposable component in some decomposition of S' into indecomposable parts. Without loss of generality we can assume, by induction assumption, that $T_1 = T_2'$ and so:

$$T = T_1' T_1 T_3' ... T_l'.$$

Further, we have

$$A_1 \sim M, B_1 \nsim M, A_1' \sim M', B_1' \nsim M.$$

This together with (21) imply

$$A_1 = A_1' = \phi, \quad \text{or } B_1 = B_1' = \phi.$$

In both situations we have

$$T_1' T_1 = T_1 T_1', T = T_1 T_1' T_3' ... T_1', T_2 ... T_K = T_1' T_3' ... T_j'.$$

By the induction assumption, one can conclude that $k = l$, and under the respective ordering, we have

$$T_1' = T_2, T_i = T_i', i = 3, ..., l. \qquad \square$$

Multiplying all undivided A-parts as well as all undivided B-parts in a decomposition (20) we obtain a canonical decomposition

$$T = C_1 C_2 ... C_r, r \geq 1 \tag{22}$$

of T. Theorem 17 implies Corollary 7.

Corollary 7: The canonical decomposition of a triad is determined uniquely.

The components C_i in decomposition (22) are called *canonical parts* of the triad T. Now, let us turn to the graphs. It is obvious that every decomposable graph G can be represented in a form $G = TG_0$ with nontrivial indecomposable G_0. We call G_0 an *indecomposable part* of G, T is called an *operator part*. Further, let (20) be the decomposition of T into indecomposable parts, then the representation

$$G = T_1 T_2 \dots T_k G_0 \tag{23}$$

is called an *operator decomposition* of G. If (22) is a canonical decomposition of T, then the representation

$$G = C_1 C_2 \dots C_r G_0 \tag{24}$$

is called a *canonical operator decomposition* of G.

For an indecomposable part G_0 we have $G = G[M]$ where M is a minimal nontrivial module of the graph G, every such module is associated with some decomposition (24). So we have Corollary 8.

Corollary 8: Every minimal nontrivial module M of a graph G determines the unique canonical operator decomposition.

5.2. (P, Q)–Decomposition

By Corollary 8, it is obvious that a graph can have several operator decompositions. But when solving a concrete problem it may be useful to admit only the decompositions whose operator parts satisfy some conditions efficient for the problem. Now we are going to present the common theory of operator decomposition with conditions for the operator part.

Let P and Q be two nonempty hereditary (with respect to induced subgraphs) classes of graphs. A graph G is called (P, Q)-split if there exists a triad (G, A, B) in T_r with $G[A] \in P$, $G[B] \in Q$ ((P, Q)-*triad*). Let us denote by (P, Q) and (P, Q)T_r the sets of all (P, Q)-split graphs and (P, Q)-triads, respectively. The set (P, Q) ((P, Q)T_r) is said to be closed hereditary class if the following conditions hold:

1. The class P is closed with respect to the join of graphs.
2. The class Q is closed with respect to the disjoint union of graphs.

The class of split graphs is the simplest example of the closed hereditary class; here P is the class of complete graphs and Q is the class of edgeless graphs. Another example, if P is the class of P_4-free graphs (cographs) and $Q = P$, then the closed hereditary class (P, Q) is the class of P_4-bipartite graphs [9].

In what follows (P, Q) is a closed hereditary class. A graph G is called (P, Q)-decomposable (decomposable on the level (P, Q)) if G can be represented in a form

$$G = TH, T \in (P,Q)T_r, H \in G_r \qquad (25)$$

Oherwise , G is (P, Q)-indecomposable. The following statement is obvious.

Proposition 4: The set $(P, Q)T_r$ is a subsemigroup in T_r if and only if it is a closed hereditary class.

If $T \in (P,Q)T_r$ and $T = T_1T_2$, then every triad T_1 and T_2 belongs to (P, Q)T_r also. Therefore if the graph H in (25) is (P, Q)-decomposable and (20) is a decomposition of the triad T into indecomposable parts, then

$$G = T_1T_2 \ldots T_kH, T_i \in (P,Q)T_r \qquad (26)$$

We call a decomposition (26) a (P, Q)-decomposition of G (or operator decomposition of the graph G on the level (P, Q)). A canonical (P, Q)-decomposition is defined analogously to a canonical decomposition of the triad T.

A module associated with a decomposition (25) is called a (P, Q)-module. A singleton (P, Q)-module as a singleton T-module of a graph G is considered to be nontrivial.

Theorem 18: Let (P, Q) be an arbitrary closed hereditary class and $G \in G_r$, then:

1. Every (P, Q)-decomposition (26) is associated with a minimal nontrivial (P, Q)-module M ($H = G\,[M]$) and determined by the module uniquely up to permutation of undivided A-parts or undivided B-parts.
2. Every minimal nontrivial (P, Q)-module M of G is associated with some (P, Q)-decomposition (26).
3. If $G \notin (P, Q)$, then (P, Q) -decomposition of G is determined uniquely up to permutation of undivided A-parts or undivided B-parts.

Proof: By the above, it remains to prove that if $G \notin (P, Q)$, then there is at most one minimal nontrivial (P, Q)-module in G. Let M_1 and M_2 be two different minimal nontrivial (P, Q)-modules in G, then

$$G = T_iH_i, T_i \in (P,Q)T_r, V(H_i) = M_i, i = 1,2,$$
$$T_1 = (G_1, A_1, B_1)$$

Further, we have $M_1 \cap M_2 = \phi$, $M_2 \subseteq A_1 \cup B_1$, $H_2 = G[M_2]$ is an induced subgraph of G_1, and $H_2 \in (P,Q)$. So we have $G = T_2H_2 \in (P,Q)$, a contradiction.

□

Corollary 9: Let $G = C_1 \ldots C_k H$ and $G' = C_1' \ldots C_1' H'$ be canonical (P, Q)-decompositions of graphs G and G', respectively. Suppose that there exists only one minimal nontrivial (P, Q)-module in G (in particular, let $G \notin (P, Q)$). Then $G \cong G'$ if and only if the following conditions hold:

1. $k = 1$.
2. $C_i \cong C_i', i = 1,\ldots,k$.
3. $H \cong H'$.

Proof: It is obvious that if conditions (1-3) hold, then $G \cong G$. Inversely, let $G \cong G'$. Without loss of generality we can consider G and G' to be equal labeled graphs. Theorem 18 implies that there exists a unique canonical (P, Q)-decomposition. □

$$G = C_1 \ldots C_1 H = G'.$$

6. Conclusion

This paper presents an optimal algorithm for finding the 1-decomposiotion of a graph and examines the connections between the 1-decomposition and well known forms of decomposition of graphs namely modular and homogeneous decomposition. The characterization of graphs which is totally decomposable by 1-decomposition is also given.

The main result of this paper is the introduction of a new form of decomposition of graph, which is a generalization of the 1-decomposition namely the (P, Q)-decomposition. This form of decomposition suggests that it is interesting from one hand to widen it's application on different class of graphs, and from the other hand to search efficient algorithms to construct this decomposition in some specious cases, as that of the 1-decomposition.

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